

ON THE ZEROth STABLE \mathbb{A}^1 -HOMOTOPY GROUP OF A SMOOTH CURVE

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ABSTRACT. We provide a cohomological interpretation of the zeroth stable \mathbb{A}^1 -homotopy group of a smooth curve over an infinite perfect field. We show that this group is isomorphic to the first Nisnevich (or Zariski) cohomology group of a certain sheaf closely related to the first Milnor-Witt K-theory sheaf. This cohomology group can be computed using an explicit Gersten-type complex. We show that if the base field is algebraically closed then the zeroth stable \mathbb{A}^1 -homotopy group of a smooth curve coincides with the zeroth Suslin homology group that was identified by Suslin and Voevodsky with a relative Picard group.

1. INTRODUCTION

A. Suslin and V. Voevodsky computed singular homology (also known as Suslin homology) groups of a smooth relative curve admitting a good compactification. For a curve over a field the result for the zeroth Suslin homology group reads as follows.

Theorem 1.1 ([SV96, Theorem 3.1]). *Let C_o be a smooth curve over a field k , let C be its compactification and $D = C - C_o$. Then there exists a natural isomorphism*

$$H_0^S(C_o) \cong \text{Pic}(C, D).$$

Here $\text{Pic}(C, D)$ is the relative Picard group, whose elements are isomorphism classes of pairs (L, ϕ) , where L is a line bundle over C and $\phi: L|_D \xrightarrow{\sim} \mathbf{1}_D$ is a trivialization of L over D . The group structure is given by tensor product. One can identify [SV96, Lemma 2.1] the relative Picard group with sheaf cohomology groups,

$$\text{Pic}(C, D) \cong H_{\text{Zar}}^1(C, (\mathbb{G}_m)_{C,D}) \cong H_{\text{Nis}}^1(C, (\mathbb{G}_m)_{C,D}) \cong H_{\text{ét}}^1(C, (\mathbb{G}_m)_{C,D}),$$

where $(\mathbb{G}_m)_{C,D} = \ker[\mathbb{G}_m \rightarrow (i_D)_*\mathbb{G}_m]$ is the sheaf of invertible functions on C trivial (equal to 1) over D . Moreover, one has the following long exact sequence [SV96, Lemma 2.3].

$$0 \rightarrow (\mathbb{G}_m)_{C,D}(C) \rightarrow \{f \in k(C) \mid f \text{ is defined and equal to 1 over } D\} \rightarrow \bigoplus_{x \in C_o^{(1)}} \mathbb{Z} \rightarrow \text{Pic}(C, D) \rightarrow 0.$$

One can regard the two terms in the middle as a Gersten-type resolution for $(\mathbb{G}_m)_{C,D}$ and the exact sequence yields that the first sheaf cohomology group of $(\mathbb{G}_m)_{C,D}$ can be computed as the first cohomology group of the corresponding Gersten-type complex. In [SV96] the authors show that $H_0^S(C_o)$ also sits in the above exact sequence, thus $H_0^S(C_o) \cong \text{Pic}(C, D)$. From our point of view the essential part of the theorem is that the Suslin homology group $H_0^S(C_o)$ is isomorphic to the sheaf cohomology group of $(\mathbb{G}_m)_{C,D}$ (and sits in the above exact sequence), while $\text{Pic}(C, D)$ gives a nice geometrical interpretation of the latter group.

The zeroth Suslin homology group of a curve can be defined as the group of morphisms in the Voevodsky's triangulated motivic category $\text{DM}(k)$ [MVW06] from the motive of the base field to the motive of the curve,

$$H_0^S(C_o) = \text{Hom}_{\text{DM}(k)}(M(\text{Spec } k), M(C_o)).$$

It looks reasonable to address the similar problem for the zeroth stable motivic homotopy group, i.e. to compute

$$\pi_0^{\mathbb{A}^1}(C_o) = \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_{\mathbb{T}}^{\infty}(\mathrm{Spec} k)_+, \Sigma_{\mathbb{T}}^{\infty}(C_o)_+),$$

where $\mathcal{SH}(k)$ is the stable motivic homotopy category [MV99, V98, Mor04]. A. Asok and C. Haesemeyer described this group for a smooth projective variety [AH11, Theorem 4.3.1]; see also Theorem 5.10 of the present paper for a slight variation of the result: for the description of

$$\mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_{\mathbb{T}}^{\infty}(\mathrm{Spec} k)_+, \Sigma_{\mathbb{G}_m}^l \Sigma_{\mathbb{T}}^{\infty} X_+)$$

with X being smooth and projective. For a curve the result looks as follows.

Theorem 1.2 ([AH11, Theorem 4.3.1] or Theorem 5.10). *Let C be a smooth projective curve over an infinite perfect field k of characteristic unequal to 2. Then there exists a natural isomorphism*

$$\pi_0^{\mathbb{A}^1}(C) \cong H_{\mathrm{Nis}}^1(C, K_1^{\mathrm{MW}} \otimes \omega_C).$$

The unramified Milnor-Witt K-theory sheaf K_n^{MW} that appears in the theorem can be defined by means of the cartesian square

$$\begin{array}{ccc} K_n^{\mathrm{MW}} & \xrightarrow{\quad} & I^n \\ \downarrow & & \downarrow \\ K_n^{\mathrm{M}} & \xrightarrow{\quad} & K_n^{\mathrm{M}}/2K_n^{\mathrm{M}} \cong I^n/I^{n+1}, \end{array}$$

where $I = \ker \left[W \xrightarrow{\mathrm{rank}} \mathbb{Z}/2\mathbb{Z} \right]$ is the fundamental ideal in the Witt ring (see [Mor12, Chapter 3]).

This description gives rise to an action of \mathbb{G}_m on K_n^{MW} : $a \in \mathbb{G}_m(U) = k[U]^*$ acts trivially on $K_n^{\mathrm{M}}(U)$ and via multiplication with the rank one quadratic form $\langle a \rangle$ on $I^n(U)$. The twisted Milnor-Witt K-theory sheaf $K_n^{\mathrm{MW}} \otimes \omega_C$ is the sheaf associated with the presheaf

$$U \mapsto K_n^{\mathrm{MW}}(U) \otimes_{\mathbb{Z}[k[U]^*]} \mathbb{Z}[\Gamma(U, \omega_C)^0],$$

where $\Gamma(U, \omega_C)^0$ is the set of nowhere vanishing sections (i.e. trivializations) of $(\omega_C)|_U$, see Definition 2.7 for the details.

The cohomology group from Theorem 1.2 also can be computed using a two term Gersten complex (Rost-Schmid complex in the notation of [Mor12, Chapter 5]),

$$(K_1^{\mathrm{MW}} \otimes \omega_C)(k(C)) \rightarrow \bigoplus_{x \in C^{(1)}} \mathrm{GW}(k(x)) \rightarrow H_{\mathrm{Nis}}^1(C, K_1^{\mathrm{MW}} \otimes \omega_C) \rightarrow 0,$$

see [AH11, Corollary 4.1.12] or Theorem 3.7 of the present paper.

A close comparison of the results of Suslin–Voevodsky and Asok–Haesemeyer combined with the isomorphisms $\mathbb{G}_m \cong K_1^{\mathrm{M}}$, $\mathbb{Z} \cong K_0^{\mathrm{M}}$ and $\mathrm{GW} \cong K_0^{\mathrm{MW}}$ immediately suggests a general statement for $\pi_0^{\mathbb{A}^1}(C_o)$ that is proved in the present paper.

Theorem 1.3 (Theorem 5.11 and Remark 5.12). *Let C_o be a smooth curve over an infinite perfect field k , let C be its compactification and $D = C - C_o$. Then there exist natural isomorphisms*

$$\pi_0^{\mathbb{A}^1}(C_o) \cong H_{\mathrm{Nis}}^1(C, (K_1^{\mathrm{MW}} \otimes \omega_C)_{C,D}) \cong H_{\mathrm{Zar}}^1(C, (K_1^{\mathrm{MW}} \otimes \omega_C)_{C,D}),$$

where

$$(K_1^{\mathrm{MW}} \otimes \omega_C)_{C,D} = \ker [K_1^{\mathrm{MW}} \otimes \omega_C \rightarrow (i_D)_*(K_1^{\mathrm{MW}} \otimes i_D^* \omega_C)]$$

for the closed embedding $i_D: D \rightarrow C$.

Moreover, there is an exact sequence

$$\{a \in (K_1^{\mathrm{MW}} \otimes \omega_C)(k(C)) \mid a \text{ is defined and trivial over } D\} \rightarrow \bigoplus_{x \in C_o^{(1)}} \mathrm{GW}(k(x)) \xrightarrow{\Theta} \pi_0^{\mathbb{A}^1}(C_o) \rightarrow 0$$

such that $\Theta(\langle 1 \rangle_y) = \Sigma_T^\infty(i_y)_+$ for a rational point $y \in C_o(k)$, the associated closed embedding $i_y: \operatorname{Spec} k \rightarrow C_o$ and $\langle 1 \rangle_y$ being the unit quadratic form belonging to the corresponding direct summand.

In the same theorem we give a similar description for a shifted in the \mathbb{G}_m -direction group,

$$\operatorname{Hom}_{\mathcal{SH}(k)}(\Sigma_T^\infty(\operatorname{Spec} k)_+, \Sigma_{\mathbb{G}_m}^l \Sigma_T^\infty(C_o)_+).$$

In the case of a projective curve $C_o = C$ we obtain precisely the description given by A. Asok and C. Haesemeyer. On the other hand, the right-hand sides in Theorems 1.1 and 1.3 are the same up to the substitution of the Milnor-Witt K-theory for the Milnor K-theory. Indeed, one can easily show that $K_1^M \otimes \omega_C \cong K_1^M$, since the action of \mathbb{G}_m on K_1^M is trivial. In particular, we have the following corollary.

Corollary 1.4. *Let C_o be a smooth curve over an algebraically closed field k , let C be its compactification and $D = C - C_o$. Then there exist natural isomorphisms*

$$\pi_0^{\mathbb{A}^1}(C_o) \cong H_0^S(C_o) \cong \operatorname{Pic}(C, D).$$

This corollary allows one to avoid the technical discussion of transfers in the proof of the following version of rigidity theorem (cf. [Ya04, Theorem 1.9] and [RØ08, Theorem 4.9]), since the pairing $\operatorname{Pic}(C, D) \times A(C_o) \rightarrow A(\operatorname{Spec} k)$ is immediate. The claim follows from divisibility of the Picard group as in loc. cit.

Theorem 1.5. *Let C_o be a smooth curve over an algebraically closed field k and let $i_0: \operatorname{Spec} k \rightarrow C_o$ and $i_1: \operatorname{Spec} k \rightarrow C_o$ be two closed points. Then for $A \in \mathcal{SH}(k)$ such that $nA = 0$ for some integer n coprime to $\operatorname{char} k$ one has*

$$i_0^* = i_1^*: A(C_o) \rightarrow A(\operatorname{Spec} k).$$

Moreover, it is well-known that the above theorem yields the following one via an argument due to A. Suslin [Sus83, the proof of Main Theorem].

Theorem 1.6 (cf. [Ya04, Theorem 1.10] and [RØ08, Theorem 4.10]). *Let K/k be an extension of algebraically closed fields, let X be a smooth variety over k and let $A \in \mathcal{SH}(k)$ be such that $nA = 0$ for some integer n coprime to $\operatorname{char} k$. Then the map*

$$A(X) \rightarrow A(X \times_{\operatorname{Spec} k} \operatorname{Spec} K)$$

is an isomorphism.

The paper is organized in the following way. The reader is assumed to be familiar with the basic notions of stable motivic homotopy theory (see [V98, MV99, Mor04]). In Section 2 we recall some properties of stable motivic homotopy sheaves and study the orientation phenomena, i.e. relate the values of stable motivic homotopy sheaves on Thom spaces of various vector bundles. In Section 3 we organize well-known facts about coniveau spectral sequence and show that the Rost-Schmid complex computes Nisnevich and Zariski cohomology of stable motivic homotopy sheaves. In Section 4 we derive a variation of Rost-Schmid complex for a pair (C, D) with C being a smooth curve and D a finite collection of closed points. In Section 5 we adopt the approach of Asok-Haesemeyer applying the Atiyah duality in order to obtain a cohomological description of the zeroth stable homotopy group. Then we apply the developed technique deriving an explicit description for the zeroth stable motivic homotopy group of \mathbb{G}_m -suspensions of a smooth projective variety and of a smooth curve. In the Appendix we provide some details on the theory of presheaves with framed transfers (see [GP14, GP15]) that are used in the present paper.

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Notation 1.7. We adopt the following notation and conventions.

- k is an infinite perfect field.
- Sm_k is the category of smooth varieties over k .
- Sm_k/X is the category of regular morphisms $Y \rightarrow X$ with Y and X being smooth varieties, i.e. Sm_k/X is the slice category over X . Note that this category is different from the category Sm_X of smooth schemes over X .
- $\mathcal{SH}(k)$ is the stable motivic homotopy category over k (see [V98, MV99, Mor04]).
- $[A, B] = \mathrm{Hom}_{\mathcal{SH}(k)}(A, B)$ for $A, B \in \mathcal{SH}(k)$.
- $\mathbb{T} = \mathbb{A}^1/(\mathbb{A}^1 - 0)$, $\mathbb{G}_m = (\mathbb{A}^1 - 0, 1)$.
- $\mathrm{Th}(E) = E/(E - z(X))$ is the Thom space of a vector bundle E over a smooth variety X with the zero section $z: X \rightarrow E$.
- $A(l) = \Sigma_{\mathbb{G}_m}^l A$ and $A[i] = \Sigma_{\mathbb{S}^1}^i A$ for $A \in \mathcal{SH}(k)$.
- $[X, A] = [\Sigma_{\mathbb{T}}^\infty X_+, A]$ for $X \in \mathrm{Sm}_k$ and $A \in \mathcal{SH}(k)$.
- $[P, A] = [\Sigma_{\mathbb{T}}^\infty P, A]$ for a pointed Nisnevich sheaf P on Sm_k and $A \in \mathcal{SH}(k)$. In particular, we write

$$[X/Y, A] = [\Sigma_{\mathbb{T}}^\infty(X/Y), A]$$

for smooth varieties $Y \subset X$. Here the varieties are treated as representable Nisnevich sheaves via Yoneda embedding.

- Essentially smooth scheme is a noetherian scheme which is the inverse limit of a filtering system with each transition morphism being an etale affine morphism of smooth varieties. For a presheaf on the category of smooth varieties we extend it to the category of essentially smooth schemes by the respective colimit.

2. HOMOTOPY SHEAVES

In this section we recall the notions of homotopy sheaves and presheaves of spectra and study the orientation phenomena for these (pre-)sheaves relating its values on Thom spaces of vector bundles.

Definition 2.1. For $A \in \mathcal{SH}(k)$ and $i, n \in \mathbb{Z}$ let $\pi_i^{\mathbb{A}^1}(A)_n(-)$ be the *homotopy presheaf* of A that is given by

$$\pi_i^{\mathbb{A}^1}(A)_n(U) = [U[i], A(n)]$$

for $U \in \mathrm{Sm}_k$. The associated Zariski sheaf is denoted $\underline{\pi}_i^{\mathbb{A}^1}(A)_n$ and referred to as a *homotopy sheaf* of A . Note that for a smooth variety X the isomorphism $\mathbb{T} \cong \mathbb{S}^1 \wedge \mathbb{G}_m$ [MV99, Lemma 2.15] identifies

$$[\Sigma_{\mathbb{T}}^n X_+, A] \cong \pi_n^{\mathbb{A}^1}(A)_{-n}(X).$$

Remark 2.2. It follows from the general theory of presheaves with framed transfers developed in [GP15] that $\underline{\pi}_i^{\mathbb{A}^1}(A)_n$ are Nisnevich sheaves (see Definition A.2 and Lemma A.3 in the current paper). Thus the given above definition of $\underline{\pi}_i^{\mathbb{A}^1}(A)_n$ coincides with the usual one, where Nisnevich sheafification is considered.

Definition 2.3. Let E be a rank r vector bundle over a smooth variety X . For $A \in \mathcal{SH}(k)$ let $\pi_i^{\mathbb{A}^1}(A)_n(-; E)$ be the presheaf on Sm_k/X given by

$$\pi_i^{\mathbb{A}^1}(A)_n(Y; E) = [\mathrm{Th}(f^*E)[i - r], A(n + r)]$$

with $f: Y \rightarrow X$ being the structure map. We denote $\underline{\pi}_i^{\mathbb{A}^1}(A)_n(-; E)$ the associated Zariski sheaf on Sm_k/X .

Remark 2.4. The shift of the indices in the above definition guarantees that a trivialization $E \cong \mathbf{1}_X^r$ induces an isomorphism of presheaves on Sm_k/X

$$\pi_i^{\mathbb{A}^1}(A)_n(-; E) \cong \pi_i^{\mathbb{A}^1}(A)_n(-).$$

Definition 2.5. Let E be a rank r vector bundle over a smooth variety X and $A \in \mathcal{SH}(k)$. For a regular morphism of smooth varieties $f: Y \rightarrow X$ we have a right action of the group of vector bundle automorphisms $\mathrm{GL}(f^*E)$ on $\pi_i^{\mathbb{A}^1}(A)_n(Y; E) = [\mathrm{Th}(f^*E)[i-r], A(n+r)]$ induced by the left action on the Thom space $\mathrm{Th}(f^*E)$.

In particular, we have a right action of $\mathrm{GL}_r(k[X])$ on $\pi_i^{\mathbb{A}^1}(A)_n(X)$ given by the identification

$$\pi_i^{\mathbb{A}^1}(A)_n(X) \cong \pi_i^{\mathbb{A}^1}(A)_n(X; \mathbf{1}_X^r)$$

combined with the left action of $\mathrm{GL}_r(k[X])$ on $\mathbf{1}_X^r$. For $\alpha \in \pi_i^{\mathbb{A}^1}(A)_n(X)$, $g \in \mathrm{GL}_r(k[X])$ we denote this action $\alpha \cdot g$.

Lemma 2.6. *Let X be a smooth variety and $x \in X$ be a (not necessary closed) point. Then for every $A \in \mathcal{SH}(k)$ we have*

- (1) $\alpha \cdot g = \alpha \cdot \det g$ for $\alpha \in \pi_i^{\mathbb{A}^1}(A)_n(\mathrm{Spec} \mathcal{O}_{X,x})$, $g \in \mathrm{GL}_r(\mathcal{O}_{X,x})$.
- (2) $\alpha \cdot a = \alpha \cdot a^{-1}$ for $\alpha \in \pi_i^{\mathbb{A}^1}(A)_n(X)$, $g \in k[X]^*$.

Proof. For the first claim recall that over a local ring every matrix of determinant one is a product of elementary transvections. Then the claim follows from the well-known fact that the action given by such matrix is homotopy trivial (see, for example, [An16, Lemma 1]).

The second claim follows from [An16, Lemma 5]. \square

Definition 2.7. Let L be a line bundle over a smooth variety X . For $A \in \mathcal{SH}(k)$ denote $\pi_i^{\mathbb{A}^1}(A)_n \otimes L$ the presheaf of abelian groups on Sm_k/X given by

$$(\pi_i^{\mathbb{A}^1}(A)_n \otimes L)(Y) = \pi_i^{\mathbb{A}^1}(A)_n(Y) \otimes_{\mathbb{Z}[k[Y]^*]} \mathbb{Z}[\Gamma(Y; f^*L)^0].$$

Here $f: Y \rightarrow X$ is the structure map, $\Gamma(Y; f^*L)^0$ is the set of nowhere vanishing global sections of f^*L (i.e. the set of trivializations of f^*L) and the action of $\mathrm{GL}_1(k[Y]) = k[Y]^*$ on $\pi_i^{\mathbb{A}^1}(A)_n(Y)$ is given in Definition 2.5. We denote $\pi_i^{\mathbb{A}^1}(A)_n \otimes L$ the associated Zariski sheaf on Sm_k/X .

Remark 2.8. It follows from Lemma 2.6 that

$$\pi_i^{\mathbb{A}^1}(A)_n \otimes L \cong \pi_i^{\mathbb{A}^1}(A)_n \otimes L^{-1}.$$

Lemma 2.9. *Let E be a vector bundle over a smooth variety X . Then for every $A \in \mathcal{SH}(k)$ there exists a canonical isomorphism of sheaves*

$$\pi_i^{\mathbb{A}^1}(A)_n(-; E) \cong \pi_i^{\mathbb{A}^1}(A)_n \otimes \det E.$$

In particular, $\pi_i^{\mathbb{A}^1}(A)_n(-; L) \cong \pi_i^{\mathbb{A}^1}(A)_n \otimes L$ for a line bundle L over $X \in \mathrm{Sm}_k$.

Proof. Put $r = \mathrm{rank} E$. For a regular morphism of smooth varieties $f: Y \rightarrow X$ consider the following homomorphisms of abelian groups.

$$\begin{array}{ccc} \pi_i^{\mathbb{A}^1}(A)_n(Y; \mathbf{1}_X^r) \times \mathbb{Z}[\mathrm{Iso}(f^*E, \mathbf{1}_Y^r)] & & \\ \phi_Y \swarrow & & \searrow \psi_Y \\ \pi_i^{\mathbb{A}^1}(A)_n(Y; E) & & \pi_i^{\mathbb{A}^1}(A)_n(Y) \otimes_{\mathbb{Z}[k[Y]^*]} \mathbb{Z}[\Gamma(Y; \det f^*E)^0] \end{array}$$

Here

- $\mathrm{Iso}(f^*E, \mathbf{1}_Y^r)$ is the set of vector bundle isomorphisms $f^*E \cong \mathbf{1}_Y^r$ (the set of trivializations of f^*E).
- For $\alpha \in \pi_i^{\mathbb{A}^1}(A)_n(Y; \mathbf{1}_X^r)$, $\theta \in \mathrm{Iso}(f^*E, \mathbf{1}_Y^r)$ we put $\phi_Y(\alpha, \theta) = \theta^*(\alpha)$ for the isomorphism $\theta^*: \pi_i^{\mathbb{A}^1}(A)_n(Y; \mathbf{1}_X^r) \xrightarrow{\cong} \pi_i^{\mathbb{A}^1}(A)_n(Y; E)$ induced by the corresponding isomorphism of Thom spaces $\mathrm{Th}(E) \xrightarrow{\cong} \mathrm{Th}(\mathbf{1}_Y^r) = \mathrm{Th}(f^*\mathbf{1}_X^r)$.

- For $\alpha \in \pi_i^{\mathbb{A}^1}(A)_n(Y; \mathbf{1}_X^r)$, $\theta \in \mathcal{I}so(f^*E, \mathbf{1}_Y^r)$ we put $\psi_Y(a, \theta) = a \otimes (\det \theta)^{-1}$. Here we identify $\pi_i^{\mathbb{A}^1}(A)_n(Y; \mathbf{1}_X^r) \cong \pi_i^{\mathbb{A}^1}(A)_n(Y)$.

These homomorphisms induce morphisms of presheaves of abelian groups on Sm_k/X ,

$$\pi_i^{\mathbb{A}^1}(A)_n(-; E) \xleftarrow{\phi} \pi_i^{\mathbb{A}^1}(A)_n(-; \mathbf{1}_X^r) \times \mathbb{Z}[\mathcal{I}so((-)^*E, \mathbf{1}_{(-)}^r)] \xrightarrow{\psi} \pi_i^{\mathbb{A}^1}(A)_n \otimes \det E.$$

Note that $\mathcal{I}so(f^*E, \mathbf{1}_Y^r)$ has a canonical left action of $\mathrm{GL}_r(k[Y])$, while

$$\pi_i^{\mathbb{A}^1}(A)_n(Y; \mathbf{1}_X^r) = \pi_i^{\mathbb{A}^1}(A)_n(Y; f^*\mathbf{1}_X^r) = \pi_i^{\mathbb{A}^1}(A)_n(Y; \mathbf{1}_Y^r)$$

has a right action of $\mathrm{GL}_r(k[Y])$ given in Definition 2.5. One easily checks that $\phi_Y(\alpha \cdot g, \theta) = \phi_Y(\alpha, g\theta)$ for $g \in \mathrm{GL}_r(k[Y])$. Moreover, Lemma 2.6 yields that for $g \in \mathrm{GL}_r(k[Y])$ the stalks of

$$\psi_Y(\alpha \cdot g, \theta) = (\alpha \cdot g) \otimes (\det \theta)^{-1}$$

and

$$\psi_Y(\alpha, g\theta) = \alpha \otimes (\det \theta)^{-1}(\det g)^{-1} = \alpha \cdot (\det g)^{-1} \otimes (\det \theta)^{-1}$$

are equal. Thus we have the induced morphisms of sheaves of abelian groups on Sm_k/X

$$\underline{\pi}_i^{\mathbb{A}^1}(A)_n(-; E) \xleftarrow{\phi} \underline{\pi}_i^{\mathbb{A}^1}(A)_n(-; \mathbf{1}_X^r) \otimes_{\mathbb{Z}[\mathrm{GL}_r(-)]} \mathbb{Z}[\mathcal{I}so((-)^*E, \mathbf{1}_{(-)}^r)] \xrightarrow{\psi} \underline{\pi}_i^{\mathbb{A}^1}(A)_n \otimes \det E.$$

One easily sees that these morphisms are stalk-wise isomorphisms, whence the claim. \square

Remark 2.10. Using a similar reasoning as above one can show that

$$\underline{\pi}_i^{\mathbb{A}^1}(A)_n \otimes (L_1 \otimes L_2^{\otimes 2}) \cong \underline{\pi}_i^{\mathbb{A}^1}(A)_n \otimes L_1$$

for line bundles L_1, L_2 over a smooth variety X .

Definition 2.11. Recall [Mor04, Section 5.2] that the *homotopy t-structure* on $\mathcal{SH}(k)$ is given by the following full subcategories:

$$\begin{aligned} \mathcal{SH}(k)_{t \geq 0} &= \{A \in \mathcal{SH}(k) \mid \underline{\pi}_i^{\mathbb{A}^1}(A)_n = 0 \text{ for } i < 0, n \in \mathbb{Z}\}, \\ \mathcal{SH}(k)_{t \leq 0} &= \{A \in \mathcal{SH}(k) \mid \underline{\pi}_i^{\mathbb{A}^1}(A)_n = 0 \text{ for } i > 0, n \in \mathbb{Z}\}. \end{aligned}$$

We denote the *heart* of the homotopy t -structure as

$$\Pi_*(k) = \mathcal{SH}(k)_{t \geq 0} \cap \mathcal{SH}(k)_{t \leq 0}.$$

For $l \in \mathbb{Z}$ we have an autoequivalence on $\Pi_*(k)$ given by

$$M \mapsto M(l) = \Sigma_{\mathbb{G}_m}^l M.$$

Put $\mathcal{SH}(k)_{t \geq n} = \mathcal{SH}(k)_{t \geq 0}[n]$, $\mathcal{SH}(k)_{t \leq n} = \mathcal{SH}(k)_{t \leq 0}[n]$. Then for every $A \in \mathcal{SH}(k)$ there exists a canonical filtration

$$\dots \rightarrow A_{t \geq n} \rightarrow A_{t \geq n-1} \rightarrow \dots \rightarrow A_{t \geq 1} \rightarrow A_{t \geq 0} \rightarrow A_{t \geq -1} \rightarrow \dots \rightarrow A$$

with $A_{t \geq n} \in \mathcal{SH}(k)_{t \geq n}$. Denote

$$H_n^t(A) = \mathrm{Cone}(A_{t \geq n+1} \rightarrow A_{t \geq n})[-n] \in \Pi_*(k).$$

Notation 2.12. Abusing the notation, for $M \in \Pi_*(k)$ we put

$$M_n(-; E) = \underline{\pi}_0^{\mathbb{A}^1}(M)_n(-; E), \quad M_n \otimes L = \underline{\pi}_0^{\mathbb{A}^1}(M)_n \otimes L \cong \underline{\pi}_0^{\mathbb{A}^1}(M)_n(-; L).$$

3. CONIVEAU SPECTRAL SEQUENCE AND ROST-SCHMID COMPLEX

In this section we recall the constructions of coniveau spectral sequence and Rost-Schmid complex (cf. [Mor12, Chapter 5]) and show that Rost-Schmid complex allows one to compute sheaf cohomology of homotopy sheaves.

Notation 3.1. For a smooth variety X and a smooth subvariety $Z \subset X$ denote

$$\Lambda_Z^X = \det N_{Z/X}$$

the determinant of the normal bundle.

Definition 3.2. Let X be a smooth variety of dimension d . Consider a sequence of open subsets

$$\emptyset = U_\alpha^{(0)} \subset U_\alpha^{(1)} \subset \dots \subset U_\alpha^{(d)} \subset U_\alpha^{(d+1)} = X$$

satisfying

- (1) $Z_\alpha^{(p)} = U_\alpha^{(p+1)} - U_\alpha^{(p)}$ is smooth and equidimensional,
- (2) $\dim Z_\alpha^{(p)} \leq d - p$.

Then for every $A \in \mathcal{SH}(k)$ there are long exact sequences

$$\dots \rightarrow [U_\alpha^{(p+1)}/U_\alpha^{(p)}, A[m]] \rightarrow [U_\alpha^{(p+1)}, A[m]] \rightarrow [U_\alpha^{(p)}, A[m]] \rightarrow [U_\alpha^{(p+1)}/U_\alpha^{(p)}, A[m+1]] \rightarrow \dots$$

Identify $U_\alpha^{(p+1)}/U_\alpha^{(p)} \cong \mathrm{Th}(N_{Z_\alpha^{(p)}/X})$ by homotopy purity theorem [MV99, Theorem 2.23]. Then the long exact sequences could be rewritten as

$$\dots \rightarrow [\mathrm{Th}(N_{Z_\alpha^{(p)}/X}), A[m]] \rightarrow [U_\alpha^{(p+1)}, A[m]] \rightarrow [U_\alpha^{(p)}, A[m]] \rightarrow [\mathrm{Th}(N_{Z_\alpha^{(p)}/X}), A[m+1]] \rightarrow \dots$$

Rewriting $[\mathrm{Th}(N_{Z_\alpha^{(p)}/X}), A[m]] = \pi_{p-m}^{\mathbb{A}^1}(A)_{-p}(Z_\alpha^{(p)}; N_{Z_\alpha^{(p)}/X})$, taking the colimit over all the considered sequences of open subsets and applying Lemma 2.9 we obtain

$$\begin{aligned} \dots \rightarrow \bigoplus_{x \in X^{(p)}} \pi_{p-m}^{\mathbb{A}^1}(A)_{-p}(x; \Lambda_x^X) &\rightarrow \mathrm{colim}_\alpha [U_\alpha^{(p+1)}, A[m]] \rightarrow \\ &\rightarrow \mathrm{colim}_\alpha [U_\alpha^{(p)}, A[m]] \rightarrow \bigoplus_{x \in X^{(p)}} \pi_{p-m-1}^{\mathbb{A}^1}(A)_{-p}(x; \Lambda_x^X) \rightarrow \dots \end{aligned}$$

Here the sum is taken over the set of points of codimension p . These long exact sequences give rise to a *coniveau spectral sequence*

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \pi_{-q}^{\mathbb{A}^1}(A)_{-p}(x; \Lambda_x^X) \Rightarrow [X, A[p+q]].$$

Let U be an open subset of X with the closed complement $Z = X - U$ of dimension d' . If one considers only the subsets $U_\alpha^{(p)}$ containing U and the corresponding sequences

$$U/U = U_\alpha^{(d-d')}/U \subset U_\alpha^{(d-d'+1)}/U \subset \dots \subset U_\alpha^{(d)}/U \subset U_\alpha^{(d+1)}/U = X/U$$

then the above spectral sequence becomes

$$E_1^{p,q} = \bigoplus_{\substack{x \in X^{(p)} \\ x \in Z}} \pi_{-q}^{\mathbb{A}^1}(A)_{-p}(x; \Lambda_x^X) \Rightarrow [X/U, A[p+q]].$$

Note that for $U = \emptyset$ this spectral sequence coincides with the previous one. Over the spectrum of a field every vector bundle is trivial, thus we have non-canonical isomorphisms

$$E_1^{p,q} = \bigoplus_{\substack{x \in X^{(p)} \\ x \in Z}} \pi_{-q}^{\mathbb{A}^1}(A)_{-p}(x; \Lambda_x^X) \simeq \bigoplus_{\substack{x \in X^{(p)} \\ x \in Z}} \pi_{-q}^{\mathbb{A}^1}(A)_{-p}(x).$$

Definition 3.3. In the notation of Definition 3.2 for $A = M \in \Pi_*(k)$ the first page of the coniveau spectral sequence is concentrated at line $q = 0$ and looks as follows:

$$\bigoplus_{\substack{x \in X^{(0)} \\ x \in Z}} M_0(x) \rightarrow \bigoplus_{\substack{x \in X^{(1)} \\ x \in Z}} M_{-1}(x; \Lambda_x^X) \rightarrow \dots \rightarrow \bigoplus_{\substack{x \in X^{(d-1)} \\ x \in Z}} M_{1-d}(x; \Lambda_x^X) \rightarrow \bigoplus_{\substack{x \in X^{(d)} \\ x \in Z}} M_{-d}(x; \Lambda_x^X).$$

We denote this complex $\mathcal{RS}_\bullet^X(X, U; M)$ and refer to it as *Rost-Schmid complex* (see [Mor12, Chapter 5]). For $U = \emptyset$ we put $\mathcal{RS}_\bullet^X(X; M) = \mathcal{RS}_\bullet^X(X, \emptyset; M)$.

Let E be a rank r vector bundle over $X \in \text{Sm}_k$ and $z: X \rightarrow E$ be the zero section. Then the shifted Rost-Schmid complex $\mathcal{RS}_\bullet^E(E, E - z(X); M(r))[r]$ looks as follows.

$$\begin{aligned} \bigoplus_{x \in X^{(0)}} M_0(x; \det E|_x) &\rightarrow \bigoplus_{x \in X^{(1)}} M_{-1}(x; \Lambda_x^X \otimes \det E|_x) \rightarrow \dots \\ &\dots \rightarrow \bigoplus_{x \in X^{(d-1)}} M_{1-d}(x; \Lambda_x^X \otimes \det E|_x) \rightarrow \bigoplus_{x \in X^{(d)}} M_{-d}(x; \Lambda_x^X \otimes \det E|_x). \end{aligned}$$

We denote this complex $\mathcal{RS}_\bullet^X(X; E; M)$.

The above construction of Rost-Schmid complex gives rise to the following complex of sheaves on the small Nisnevich site of X :

$$\begin{aligned} \bigoplus_{x \in X^{(0)}} (i_x)_* M_0(-; \det E|_x) &\rightarrow \bigoplus_{x \in X^{(1)}} (i_x)_* M_{-1}(-; \Lambda_x^X \otimes \det E|_x) \rightarrow \dots \\ &\dots \rightarrow \bigoplus_{x \in X^{(d-1)}} (i_x)_* M_{1-d}(-; \Lambda_x^X \otimes \det E|_x) \rightarrow \bigoplus_{x \in X^{(d)}} (i_x)_* M_{-d}(-; \Lambda_x^X \otimes \det E|_x). \end{aligned}$$

Here $(i_x)_* M_p(-; \Lambda_x^X \otimes \det E|_x)$ is the Nisnevich direct image sheaf for the embedding $i_x: x \rightarrow X$. We denote this complex $\mathcal{RS}_\bullet^X(-; E; M)$. Note that it follows from the construction that

$$\mathcal{RS}_\bullet^X(-; E; M) = \mathcal{RS}_\bullet^X(-; \det E; M).$$

Lemma 3.4. *Let X be a smooth variety and $U \subset X$ be an open subset. Then for every $M \in \Pi_*(k)$ there are canonical isomorphisms*

$$[X/U, M[m]] \xrightarrow{\cong} H^m(\mathcal{RS}_\bullet^X(X, U; M)).$$

In particular, for a rank r vector bundle E over $X \in \text{Sm}_k$ we have

$$[\text{Th}(E), M(r)[r+m]] \xrightarrow{\cong} H^m(\mathcal{RS}_\bullet^X(X; \det E; M)).$$

Proof. For $A = M$ the first page of the coniveau spectral sequence is concentrated at one line which by definition coincides with $\mathcal{RS}_\bullet^X(X, U; M)$. \square

Lemma 3.5. *Let X be a smooth variety and $x \in X$ be a (not necessary closed) point. Denote $i: x \rightarrow X$ the corresponding embedding. Then for a sheaf \mathcal{F} on the small Nisnevich site x_{Nis} and a closed subset $Z \subset X$ we have*

$$H_{\text{Zar}, Z}^m(X; i_* \mathcal{F}) = H_{\text{Nis}, Z}^m(X; i_* \mathcal{F}) = 0$$

for $m > 0$. Here $i_ \mathcal{F}$ is the direct image sheaf in Nisnevich topology.*

Proof. Follows from the Grothendieck spectral sequence for the composition of functors $\Gamma_Z(-) \circ i_*$ and exactness of i_* and $\Gamma_Z(-) \circ i_*$. \square

Lemma 3.6. *Let L be a line bundle over $X \in \text{Sm}_k$. Then for $M \in \Pi_*(k)$ the complex*

$$M_0(-; L) \rightarrow \mathcal{RS}_\bullet^X(-; L; M)$$

is a resolution in both Zariski and Nisnevich topologies on the corresponding small site of X . Here

$$M_0(-; L) \rightarrow \mathcal{RS}_0^X(-; L; M) = \bigoplus_{x \in X^{(0)}} (i_x)_* M_0(-; L|_x)$$

is induced by restriction homomorphisms

$$M_0(U; L) \rightarrow \bigoplus_{x \in U^{(0)}} M_0(x; L|_x)$$

for étale $U \rightarrow X$.

Proof. Lemma 3.4 yields that for a local scheme $W = \operatorname{Spec} \mathcal{O}_{X,x}$ (or $W = \operatorname{Spec} \mathcal{O}_{X,x}^h$) we have

$$H^m(\mathcal{RS}_\bullet^X(W; L; M)) \cong [\operatorname{Th}(L|_W), M(1)[m+1]] \cong \underline{\pi}_m^{\mathbb{A}^1}(M)_0(W).$$

The last group is trivial for $m \neq 0$ since $M \in \Pi_*(k)$, whence the claim. \square

Theorem 3.7 (cf. [Mor12, Corollary 5.43]). *Let E be a rank r vector bundle over a smooth variety X and let $U \subset X$ be an open subset. Put $Z = X - U$. Then for every $M \in \Pi_*(k)$ there exist canonical isomorphisms*

- (1) $[X/U, M[m]] \cong H^m(\mathcal{RS}_\bullet^X(X, U; M)) \cong H_{\operatorname{Nis}, Z}^m(X; M_0) \cong H_{\operatorname{Zar}, Z}^m(X; M_0),$
- (2) $[\operatorname{Th}(E), M(r)[r+m]] \cong H^m(\mathcal{RS}_\bullet^X(X; \det E; M)) \cong H_{\operatorname{Nis}}^m(X; M_0 \otimes \det E) \cong H_{\operatorname{Zar}}^m(X; M_0 \otimes \det E).$

In particular,

$$[X, M] \cong M_0(X) = \underline{\pi}_0^{\mathbb{A}^1}(M)_0(X), \quad [\operatorname{Th}(E), M(r)[r]] \cong M_0(X; E) = \underline{\pi}_0^{\mathbb{A}^1}(M)_0(X; E),$$

i.e. $\pi_0^{\mathbb{A}^1}(M)_0$ and $\pi_0^{\mathbb{A}^1}(M)_0(-; E)$ are sheaves.

Proof. Follows from Lemmas 3.4, 3.5 and 3.6. \square

Remark 3.8. It follows from the above theorem and Remark 2.10 that if $M \in \Pi_*(k)$ is a commutative monoid, then the ring cohomology theory represented by M is SL^c -oriented in the sense of [PW10, Definition 3.3]. In particular,

$$\pi_i^{\mathbb{A}^1}(M)_n(X; E) \cong \pi_i^{\mathbb{A}^1}(M)_n(X; \det E).$$

See also [AH11, Theorem 4.2.7] for the case of $M = H_0^t(\mathbb{S})$.

Lemma 3.9. *Let X be a smooth variety of dimension d , $\rho: \tilde{X} \rightarrow X$ be a Zariski locally trivial \mathbb{A}^s -fibration, E be a rank r vector bundle over \tilde{X} , L be a line bundle over X and $\theta: \det E \cong \rho^* L$ be an isomorphism. Then for every $M \in \Pi_*(k)$ there exists an isomorphism*

$$\Theta: H^d(\mathcal{RS}_\bullet^X(X; L; M)) \xrightarrow{\sim} [\operatorname{Th}(E), M(r)[r+d]]$$

such that for every rational point $y \in X(k)$ the following diagram commutes.

$$\begin{array}{ccccc} \pi_0^{\mathbb{A}^1}(M)_{-d}(\tilde{y}; \Lambda_{\tilde{y}}^E) & \xrightarrow[\simeq]{\phi} & M_{-d}(y; \Lambda_y^X \otimes L_y) & \xrightarrow{i_y} & H^n(\mathcal{RS}_\bullet^X(X; L; M)) \\ \Psi \downarrow \cong & & & & \downarrow \cong \Theta \\ \pi_0^{\mathbb{A}^1}(M)_{-d}(\tilde{y}; N_{\tilde{y}/E}) & \xrightarrow[\simeq]{d} & [E/(E - \tilde{y}), M(r)[r+d]] & \xrightarrow{Q} & [\operatorname{Th}(E), M(r)[r+d]] \end{array}$$

Here

- $\tilde{y} = z \circ \rho^{-1}(y) \subset E$ for the zero section $z: \tilde{X} \rightarrow E$,
- i_y is induced by the inclusion $M_{-d}(y; \Lambda_y^X \otimes L_y) \rightarrow \bigoplus_{x \in X^{(d)}} M_{-d}(x; \Lambda_x^X \otimes L_x)$ to the summand corresponding to the point y ,

- Q is induced by the quotient map $\mathrm{Th}(E) \rightarrow E/(E - \tilde{y})$,
- d is induced by the homotopy purity theorem,
- Ψ is given by Lemma 2.9,
- ϕ is induced by projection $\rho: \tilde{y} \rightarrow y$ and isomorphisms $\Lambda_{\tilde{y}}^E \cong \Lambda_{\tilde{y}}^{\tilde{X}} \otimes \det E|_{\tilde{y}} \cong \rho^*(\Lambda_y^X \otimes L_y)$.

Proof. Consider a sequence of open subsets

$$\emptyset = U_\alpha^{(0)} \subset U_\alpha^{(1)} \subset \dots \subset U_\alpha^{(d)} \subset U_\alpha^{(d+1)} = X$$

as in Definition 3.2. Put $\tilde{U}_\alpha^{(p)} = \rho^{-1}(U_\alpha^{(p)})$ and $\tilde{W}_\alpha^{(p)} = \pi^{-1}(\tilde{U}_\alpha^{(p)}) \cup (E - z(\tilde{X}))$ for the canonical projection $\pi: E \rightarrow \tilde{X}$ and the zero section $z: \tilde{X} \rightarrow E$. We have a sequence of open subsets

$$E - z(\tilde{X}) = \tilde{W}_\alpha^{(0)} \subset \tilde{W}_\alpha^{(1)} \subset \dots \subset \tilde{W}_\alpha^{(d)} \subset \tilde{W}_\alpha^{(d+1)} = E.$$

For $M \in \Pi_*(k)$ we obtain a family of long exact sequences

$$\begin{aligned} \dots \rightarrow [\mathrm{Th}(N_{\tilde{Z}_\alpha^{(p)}/E}), M(r)[m]] &\rightarrow [\tilde{W}_\alpha^{(p+1)}/\tilde{W}_\alpha^{(0)}, M(r)[m]] \rightarrow \\ &\rightarrow [\tilde{W}_\alpha^{(p)}/\tilde{W}_\alpha^{(0)}, M(r)[m]] \rightarrow [\mathrm{Th}(N_{\tilde{Z}_\alpha^{(p)}/E}), M(r)[m+1]] \rightarrow \dots \end{aligned}$$

where $\tilde{Z}_\alpha^{(p)} = z \circ \rho^{-1}(Z_\alpha^{(p)})$ with $Z_\alpha^{(p)} = U_\alpha^{(p+1)} - U_\alpha^{(p)}$.

Take the colimit over the families of $U_\alpha^{(p)}$. The resulting long exact sequences can be organized in the following version of the coniveau spectral sequence (cf. Definition 3.2):

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \pi_{r-q}^{\mathbb{A}^1}(M)_{-p}(\tilde{x}; N_{\tilde{x}/E}) \Rightarrow [\mathrm{Th}(E), M(r)[p+q]],$$

where $\tilde{x} = z \circ \rho^{-1}(x) \cong \mathbb{A}_{k(x)}^s$. Since $M \in \Pi_*(k)$ and every vector bundle over an affine space is trivial, this page is concentrated at line $q = r$ and looks in the following way.

$$\begin{aligned} \bigoplus_{x \in X^{(0)}} M_0(\tilde{x}; N_{\tilde{x}/E}) &\rightarrow \bigoplus_{x \in X^{(1)}} M_{-1}(\tilde{x}; N_{\tilde{x}/E}) \rightarrow \dots \\ &\dots \rightarrow \bigoplus_{x \in X^{(d-1)}} M_{1-d}(\tilde{x}; N_{\tilde{x}/E}) \rightarrow \bigoplus_{x \in X^{(d)}} M_{-d}(\tilde{x}; N_{\tilde{x}/E}). \end{aligned}$$

We have $\det N_{\tilde{x}/E} \cong \rho^* \Lambda_x^X \otimes \det E|_{\tilde{x}} \cong \rho^*(\Lambda_x^X \otimes L|_x)$, thus we may rewrite this complex as

$$\begin{aligned} \bigoplus_{x \in X^{(0)}} M_0(\tilde{x}; \rho^* L|_x) &\rightarrow \bigoplus_{x \in X^{(1)}} M_{-1}(\tilde{x}; \rho^*(\Lambda_x^X \otimes L|_x)) \rightarrow \dots \\ &\dots \rightarrow \bigoplus_{x \in X^{(d-1)}} M_{1-d}(\tilde{x}; \rho^*(\Lambda_x^X \otimes L|_x)) \rightarrow \bigoplus_{x \in X^{(d)}} M_{-d}(\tilde{x}; \rho^*(\Lambda_x^X \otimes L|_x)). \end{aligned}$$

Projection $\rho: \tilde{x} \rightarrow x$ is a homotopy equivalence, thus we may further rewrite it as

$$\begin{aligned} \bigoplus_{x \in X^{(0)}} M_0(x; L|_x) &\rightarrow \bigoplus_{x \in X^{(1)}} M_{-1}(x; \Lambda_x^X \otimes L|_x) \rightarrow \dots \\ &\dots \rightarrow \bigoplus_{x \in X^{(d-1)}} M_{1-d}(x; \Lambda_x^X \otimes L|_x) \rightarrow \bigoplus_{x \in X^{(d)}} M_{-d}(x; \Lambda_x^X \otimes L|_x). \end{aligned}$$

One can easily check that this complex coincides with $\mathcal{RS}_\bullet^X(X; L; M)$. The claim follows. \square

4. ROST-SCHMID COMPLEX FOR C/D

In this section we present a variant of Rost-Schmid complex for C/D with C being a smooth curve and $D \subset C^{(1)}$ being a finite collection of closed points. This complex, roughly speaking, is the cone for an appropriate morphism from Rost-Schmid complex of C to Rost-Schmid complex of D .

Definition 4.1. Let $\pi: E \rightarrow C$ be a rank r vector bundle over a smooth curve C with the zero section $z: C \rightarrow E$. Let $D \subset C^{(1)}$ be a finite collection of closed points. For an open subset $D \subset U_\alpha \subset C$ put $U_\alpha^E = \pi^{-1}(U_\alpha) \cup (E - z(C))$. Then for every $M \in \Pi_*(k)$ there is a long exact sequence

$$\begin{aligned} \dots \rightarrow [E/U_\alpha^E, M(r)[r+m]] &\rightarrow [\mathrm{Th}(E), M(r)[r+m]] \rightarrow \\ &\rightarrow [U_\alpha^E/(E - z(C)), M(r)[r+m]] \rightarrow [E/U_\alpha^E, M(r)[r+m+1]] \rightarrow \dots \end{aligned}$$

Taking the colimit over U_α and applying Lemma 2.9 we obtain a long exact sequence

$$\begin{aligned} \dots \rightarrow \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} \pi_{-m+1}^{\mathbb{A}^1}(M)_{-1}(x; \Lambda_x^C \otimes \det E|_x) &\rightarrow [\mathrm{Th}(E), M(r)[r+m]] \rightarrow \\ &\rightarrow \pi_{-m}^{\mathbb{A}^1}(M)_0(\mathrm{Spec} \mathcal{O}_{C,D}; \det E) \xrightarrow{\partial} \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} \pi_{-m}^{\mathbb{A}^1}(M)_{-1}(x; \Lambda_x^C \otimes \det E|_x) \rightarrow \dots \end{aligned}$$

The corresponding version of Rost-Schmid complex consists of two terms:

$$M_0(\mathrm{Spec} \mathcal{O}_{C,D}; \det E) \xrightarrow{\partial} \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} M_{-1}(x; \Lambda_x^C \otimes \det E|_x).$$

We consider the following modified version of Rost-Schmid complex,

$$M_0(\mathrm{Spec} \mathcal{O}_{C,D}; \det E) \xrightarrow{(\partial, i_D^*)} \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} M_{-1}(x; \Lambda_x^C \otimes \det E|_x) \oplus \bigoplus_{x \in D} M_0(x; \det E|_x),$$

where i_D^* is the restriction morphism for the embedding $i_D: D \rightarrow \mathrm{Spec} \mathcal{O}_{C,D}$. This complex is denoted $\mathcal{RS}_\bullet^C(C, D; E; M)$. Note that

$$\mathcal{RS}_\bullet^C(C, D; E; M) = \mathcal{RS}_\bullet^C(C, D; \det E; M).$$

Lemma 4.2. Let $\pi: E \rightarrow C$ be a rank r vector bundle over a smooth curve C with the zero section $z: C \rightarrow E$. Let $D \subset C^{(1)}$ be a finite collection of closed points. Then for $M \in \Pi_*(k)$ and $m \in \mathbb{Z}$ there are canonical isomorphisms

$$[\mathrm{Th}(E)/\mathrm{Th}(E|_D), M(r)[r+m]] \cong H^m(\mathcal{RS}_\bullet^C(C, D; \det E; M)).$$

Proof. It follows from the discussion in Definition 4.1 that there is a long exact sequence

$$\begin{aligned} \dots \rightarrow \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} \pi_{-m+1}^{\mathbb{A}^1}(M)_{-1}(x; \Lambda_x^C \otimes \det E|_x) &\rightarrow [\mathrm{Th}(E), M(r)[r+m]] \rightarrow \\ &\rightarrow \pi_{-m}^{\mathbb{A}^1}(M)_0(\mathrm{Spec} \mathcal{O}_{C,D}; \det E) \xrightarrow{\partial} \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} \pi_{-m}^{\mathbb{A}^1}(M)_{-1}(x; \Lambda_x^C \otimes \det E|_x) \rightarrow \dots \end{aligned}$$

Note that $\pi_{-m}^{\mathbb{A}^1}(M)_{-1}(x; \Lambda_x^C \otimes \det E|_x) = 0$ for $m \neq 0$ since $M \in \Pi_*(k)$. Moreover,

$$\pi_{-m}^{\mathbb{A}^1}(M)_0(\mathrm{Spec} \mathcal{O}_{C,D}; \det E) \cong H^m(\mathcal{RS}_\bullet^C(\mathrm{Spec} \mathcal{O}_{C,D}; E; M)) \cong H_{\mathrm{Nis}}^m(\mathrm{Spec} \mathcal{O}_{C,D}; M_0 \otimes \det E)$$

by Theorem 3.7. The cohomology of Rost-Schmid complex vanishes for $m \neq 0, 1$ since the complex consists of two terms. The Nisnevich cohomology group vanishes for $m = 1$ by Lemma A.4. Then $\pi_{-m}^{\mathbb{A}^1}(M)_0(\text{Spec } \mathcal{O}_{C,D}; \det E) = 0$ for $m \neq 0$. Thus we have an exact sequence

$$0 \rightarrow [\text{Th}(E), M(r)[r]] \rightarrow M_0(\text{Spec } \mathcal{O}_{C,D}; \det E) \xrightarrow{\partial} \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} M_{-1}(x; \Lambda_x^C \otimes \det E|_x) \rightarrow [\text{Th}(E), M(r)[r+1]] \rightarrow 0$$

and $[\text{Th}(E), M(r)[r+m]] = 0$ for $m \neq 0, 1$.

The natural embedding $\text{Th}(E|_D) \rightarrow \text{Th}(E)$ gives rise to a long exact sequence

$$\begin{aligned} \dots \rightarrow [\text{Th}(E)/\text{Th}(E|_D), M(r)[r+m]] &\rightarrow [\text{Th}(E), M(r)[r+m]] \rightarrow \\ &\rightarrow [\text{Th}(E|_D), M(r)[r+m]] \rightarrow [\text{Th}(E)/\text{Th}(E|_D), M(r)[r+m+1]] \rightarrow \dots \end{aligned}$$

By the above and since $[\text{Th}(E|_D), M(r)[r+m]] = \pi_{-m}^{\mathbb{A}^1}(M)_0(D; E|_D)$ vanishes for $m \neq 0$ we have $[\text{Th}(E)/\text{Th}(E|_D), M(r)[r+m]] = 0$ for $m \neq 0, 1$.

For $m = 0$ consider the following diagram.

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & [\text{Th}(E)/\text{Th}(E|_D), M(r)[r]] & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & [\text{Th}(E), M(r)[r]] & \longrightarrow & M_0(\text{Spec } \mathcal{O}_{C,D}; \det E) & \xrightarrow{\partial} & \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} M_{-1}(x; \Lambda_x^C \otimes \det E|_x) \\ & & \downarrow & \nearrow i_D^* & & & \\ & & M_0(D; \det E|_D) & & & & \end{array}$$

Here the vertical exact sequence is the one associated to the embedding $\text{Th}(E|_D) \rightarrow \text{Th}(E)$ while the horizontal exact sequence is a part of the coniveau one. The triangle commutes since all the involved morphisms are given by restriction and Lemma 2.9. The isomorphism

$$[\text{Th}(E)/\text{Th}(E|_D), M(r)[r]] \cong H^0(\mathcal{RS}_{\bullet}^C(C, D; \det E; M))$$

follows via diagram chase.

For $m = 1$ consider the following commutative diagram.

$$\begin{array}{ccc} [\text{Th}(E), M(r)[r]] & \xrightarrow{=} & [\text{Th}(E), M(r)[r]] \\ \downarrow & & \downarrow \\ M_0(\text{Spec } \mathcal{O}_{C,D}; \det E) & \xrightarrow{i_D^*} & M_0(D; \det E|_D) \\ \partial \downarrow & & \downarrow \\ \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} M_{-1}(x; \Lambda_x^C \otimes \det E|_x) & \longrightarrow & [\text{Th}(E)/\text{Th}(E|_D), M(r)[r+1]] \\ \downarrow & & \downarrow \\ [\text{Th}(E), M(r)[r+1]] & \xrightarrow{=} & [\text{Th}(E), M(r)[r+1]] \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Here the exact sequence in the first row is a part of the coniveau sequence while the exact sequence in the second row is the one associated to the embedding $\mathrm{Th}(E|_D) \rightarrow \mathrm{Th}(E)$. The horizontal morphisms are induced by the following morphism of triangles (to be more precise one needs to take a limit over U_α as in Definition 4.1)

$$\begin{array}{ccccccc} \mathrm{Th}(E|_D) & \longrightarrow & \mathrm{Th}(E) & \longrightarrow & \mathrm{Th}(E)/\mathrm{Th}(E|_D) & \longrightarrow & \mathrm{Th}(E|_D)[1] \\ \downarrow i_D & & \downarrow = & & \downarrow & & \downarrow \\ \mathrm{Th}(E|_{\mathrm{Spec} \mathcal{O}_{C,D}}) & \longrightarrow & \mathrm{Th}(E) & \longrightarrow & \mathrm{Th}(E)/\mathrm{Th}(E|_{\mathrm{Spec} \mathcal{O}_{C,D}}) & \longrightarrow & \mathrm{Th}(E|_{\mathrm{Spec} \mathcal{O}_{C,D}})[1] \end{array}$$

The claim follows by diagram chase. \square

Definition 4.3. Let L be a line bundle over a smooth curve C and $D \subset C^{(1)}$ be a finite collection of closed points. For $M \in \Pi_*(k)$ consider the following sheaves on the small Nisnevich site of C .

- Let $(M_0)_C^D(-; L)$ be the sheaf given by

$$(M_0)_C^D(U; L) = M_0(\mathrm{Spec} \mathcal{O}_{U, D_U}; L)$$

for an etale morphism $U \rightarrow C$ and $D_U = U \times_C D$. It follows from Lemma 3.4 that this sheaf coincides with the kernel

$$(M_0)_C^D(-; L) = \ker \left[\bigoplus_{x \in C^{(0)}} (i_x)_* M_0(-; L|_x) \xrightarrow{\partial} \bigoplus_{x \in D} (i_x)_* M_{-1}(-; \Lambda_x^C \otimes L|_x) \right].$$

Here on the right-hand side we have the constructed in Definition 3.3 Rost-Schmid complex of sheaves $\mathcal{RS}_\bullet^{\mathrm{Spec} \mathcal{O}_{C,D}}(-; L|_{\mathrm{Spec} \mathcal{O}_{C,D}}; M)$ pushed forward to C .

- Let $(M_0)_{C,D}(-; L)$ be the sheaf given by

$$(M_0)_{C,D}(U; L) = \ker \left[M_0(U; L) \xrightarrow{i_{D_U}^*} M_0(D_U; L) \right]$$

for an etale morphism $U \rightarrow C$, $D_U = U \times_C D$ and the natural embedding $i_{D_U}: D_U \rightarrow U$.

In other words, $(M_0)_{C,D}(-; L) = \ker \left[M_0(-; L) \xrightarrow{i_D^*} (i_D)_* M_0(-; L|_D) \right]$.

Note that by Lemma 2.9 we have a canonical isomorphism

$$(M_0)_{C,D}(-; L) \cong (M_0 \otimes L)_{C,D} = \ker \left[M_0 \otimes L \xrightarrow{i_D^*} (i_D)_*(M_0 \otimes L|_D) \right].$$

The discussion in Definition 4.1 gives rise to the following two term complex of sheaves on the small Nisnevich site of C .

$$(M_0)_C^D(-; L) \xrightarrow{(\partial, i_D^*)} \bigoplus_{\substack{x \in C^{(1)} \\ x \notin D}} (i_x)_* M_{-1}(-; \Lambda_x^C \otimes L|_x) \oplus \bigoplus_{x \in D} (i_x)_* M_0(-; L|_x)$$

Here $(i_x)_*$ is the direct image in Nisnevich topology. This complex is denoted $\mathcal{RS}_\bullet^C(-, D; E; M)$.

Lemma 4.4. Let L be a line bundle over a smooth curve C and $D \subset C^{(1)}$ be a finite collection of closed points. Then for $M \in \Pi_*(k)$ and $m > 0$ we have

$$H_{\mathrm{Nis}}^m(C, (M_0)_C^D(-; L)) = H_{\mathrm{Zar}}^m(C, (M_0)_C^D(-; L)) = 0.$$

Proof. Let $U \rightarrow C$ be an etale morphism and put $D_U = U \times_C D$. We claim that the sequence

$$0 \rightarrow (M_0)_C^D(U; L) \rightarrow \bigoplus_{x \in U^{(0)}} M_0(x; L|_x) \xrightarrow{\partial} \bigoplus_{x \in D_U} M_{-1}(x; \Lambda_x^U \otimes L|_x) \rightarrow 0$$

that arises from Definition 4.3 is exact. Indeed, by the definition $\ker \partial = (M_0)_C^D(U; L)$ while by Lemma 3.4 we have $\operatorname{coker} \partial \cong H_{\text{Nis}}^1(\operatorname{Spec} \mathcal{O}_{U, D_U}; M_0 \otimes L)$ and this group vanishes by Lemma A.4.

Hence

$$0 \rightarrow (M_0)_C^D(-; L) \rightarrow \bigoplus_{x \in C^{(0)}} (i_x)_* M_0(-; L|_x) \xrightarrow{\partial} \bigoplus_{x \in D} (i_x)_* M_{-1}(-; \Lambda_x^C \otimes L|_x) \rightarrow 0$$

is an exact sequence of sheaves. The claim follows from the associated long exact sequence of cohomology groups, Lemma 3.5 and the aforementioned surjectivity on global sections. \square

Lemma 4.5. *Let L be a line bundle over a smooth curve C and $D \subset C^{(1)}$ be a finite collection of closed points. Then for $M \in \Pi_*(k)$ the complex*

$$(M_0)_{C,D}(-; L) \rightarrow \mathcal{RS}_\bullet^C(-, D; L; M)$$

is a resolution in Nisnevich topology on the small site of C . Here the morphism

$$(M_0)_{C,D}(-; L) \rightarrow \mathcal{RS}_0^C(-, D; L; M) = (M_0)_C^D(-; L)$$

is induced by restriction homomorphisms

$$M_0(U; L) \rightarrow M_0(\operatorname{Spec} \mathcal{O}_{U, D_U}; L) = (M_0)_C^D(U; L),$$

where $U \rightarrow C$ is étale and $D_U = U \times_C D$.

Moreover, if for every closed point $x \in C$ the restriction homomorphism $M_0(\operatorname{Spec} \mathcal{O}_{C,x}) \rightarrow M_0(x)$ is surjective, then the complex is a resolution in Zariski topology.

Proof. For a local scheme $W = \operatorname{Spec} \mathcal{O}_{C,x}$ (or $W = \operatorname{Spec} \mathcal{O}_{C,x}^h$) and $D_W = W \times_C D$ Lemma 4.2 yields an isomorphism

$$H^m(\mathcal{RS}_\bullet^C(W, D; L; M)) \cong [\operatorname{Th}(L|_W) / \operatorname{Th}(L|_{D_W}), M(1)[m+1]].$$

The embedding $\operatorname{Th}(L|_{D_W}) \rightarrow \operatorname{Th}(L|_W)$ gives rise to an exact sequence

$$\begin{aligned} 0 \rightarrow [\operatorname{Th}(L|_W) / \operatorname{Th}(L|_{D_W}), M(1)[1]] &\rightarrow [\operatorname{Th}(L|_W), M(1)[1]] \rightarrow \\ &\rightarrow [\operatorname{Th}(L|_{D_W}), M(1)[1]] \rightarrow [\operatorname{Th}(L|_W) / \operatorname{Th}(L|_{D_W}), M(1)[2]] \rightarrow 0 \end{aligned}$$

Here all the other terms of the long exact sequence vanish since $M \in \Pi_*(k)$. Thus

$$H^0(\mathcal{RS}_\bullet^C(W, D; L; M)) \cong (M_0)_{C,D}(W)$$

and one can easily check that this isomorphism is compatible with the restriction homomorphism from the statement of the lemma.

It remains to show that $[\operatorname{Th}(L|_W), M(1)[1]] \rightarrow [\operatorname{Th}(L|_{D_W}), M(1)[1]]$ is surjective. Trivializing $L|_W$ we see that it is sufficient to show that the restriction homomorphism

$$M_0(W) \rightarrow M_0(D_W).$$

is surjective. The case when $D_W = \emptyset$ is trivial. Thus we may assume that $D_W = x$ is the closed point of $W = \operatorname{Spec} \mathcal{O}_{C,x}$ (or $W = \operatorname{Spec} \mathcal{O}_{C,x}^h$). The claim for the Zariski topology ($W = \operatorname{Spec} \mathcal{O}_{C,x}$) follows from the assumption. For the Nisnevich topology one notices that $\operatorname{Spec} \mathcal{O}_{C,x}^h \cong \operatorname{Spec}(\mathbb{A}_{k(x)}^1)_{(0)}^h$, thus the embedding $x \rightarrow \operatorname{Spec} \mathcal{O}_{C,x}^h$ has a section. \square

Theorem 4.6. *Let E be a rank r vector bundle over a smooth curve C and let $D \subset C^{(1)}$ be a finite collection of closed points. Then for $M \in \Pi_*(k)$ there exist natural isomorphisms*

$$[\operatorname{Th}(E) / \operatorname{Th}(E|_D), M(r)[r+m]] \cong H^m(\mathcal{RS}_\bullet^C(C, D; \det E; M)) \cong H_{\text{Nis}}^m(X; (M_0 \otimes \det E)_{C,D}).$$

Moreover, if for every closed point $x \in C$ the restriction homomorphism $M_0(\operatorname{Spec} \mathcal{O}_{C,x}) \rightarrow M_0(x)$ is surjective, then

$$H_{\text{Nis}}^m(X; (M_0 \otimes \det E)_{C,D}) \cong H_{\text{Zar}}^m(X; (M_0 \otimes \det E)_{C,D}).$$

Proof. The first isomorphism follows from Lemma 4.2. The other isomorphisms follow the fact that $\mathcal{RS}_\bullet^C(-, D; \det E; M)$ provides an acyclic resolution for $(M_0 \otimes \det E)_{C,D}$ by Lemmas 3.5, 4.4 and 4.5. \square

Lemma 4.7. *Let C be a smooth curve, $D \subset C^{(1)}$ be a finite collection of closed points, $\rho: \tilde{C} \rightarrow C$ be a Zariski locally trivial \mathbb{A}^s -fibration, E be a rank r vector bundle over \tilde{C} , L be a line bundle over C and $\theta: \det E \cong \rho^* L$ be an isomorphism. Put $\tilde{D} = \rho^{-1}(D)$. Then for every $M \in \Pi_*(k)$ there exists an isomorphism*

$$\Theta: H^1(\mathcal{RS}_\bullet^C(C, D; L; M)) \xrightarrow{\cong} [\mathrm{Th}(E)/\mathrm{Th}(E|_{\tilde{D}}), M(r)[r+1]]$$

such that for every rational point $y \in C(k)$, $y \notin D$, the following diagram commutes.

$$\begin{array}{ccccc} \pi_0^{\mathbb{A}^1}(M)_{-1}(\tilde{y}; \Lambda_y^E) & \xrightarrow[\cong]{\phi} & M_{-1}(y; \Lambda_y^C \otimes L_y) & \xrightarrow{i_y} & H^1(\mathcal{RS}_\bullet^C(C, D; L; M)) \\ \Psi \downarrow \cong & & & & \downarrow \cong \Theta \\ \pi_0^{\mathbb{A}^1}(M)_{-1}(\tilde{y}; N_{\tilde{y}/E}) & \xrightarrow[\cong]{d} & [E/(E - \tilde{y}), M(r)[r+1]] & \xrightarrow{Q} & [\mathrm{Th}(E)/\mathrm{Th}(E|_{\tilde{D}}), M(r)[r+1]] \end{array}$$

Here

- $\tilde{y} = z \circ \rho^{-1}(y) \subset E$ for the zero section $z: \tilde{C} \rightarrow E$,
- i_y is induced by the inclusion $M_{-1}(y; \Lambda_y^C \otimes L_y) \rightarrow \bigoplus_{\substack{x \in C^{(1)}, \\ x \notin D}} M_{-1}(x; \Lambda_x^C \otimes L_x)$ to the summand corresponding to the point y ,
- Q is induced by the quotient $\mathrm{Th}(E)/\mathrm{Th}(E|_{\tilde{D}}) \rightarrow E/(E - \tilde{y})$,
- d is induced by the homotopy purity theorem,
- Ψ is given by Lemma 2.9,
- ϕ is induced by projection $\rho: \tilde{y} \rightarrow y$ and isomorphisms $\Lambda_{\tilde{y}}^E \cong \Lambda_{\tilde{y}}^{\tilde{C}} \otimes \det E|_{\tilde{y}} \cong \rho^*(\Lambda_y^C \otimes L_y)$.

Proof. The proof is similar to the one given for Lemma 3.9. \square

5. THE DUAL SPECTRUM AND THE ZEROth STABLE HOMOTOPY GROUP

In this section we apply the Atiyah duality, basic properties of homotopy t -structure and developed techniques in order to obtain a cohomological description of the zeroth stable motivic homotopy groups.

Definition 5.1. For $A \in \mathcal{SH}(k)$ denote $A^\vee = \underline{\mathrm{Hom}}_{\mathcal{SH}(k)}(A, \mathbb{S})$ the *dual spectrum*.

Definition 5.2. Let X be a smooth variety. By Jouanolou device (see [J73], [We89, § 4]) there is a morphism $\rho: \tilde{X} \rightarrow X$ such that \tilde{X} is affine and ρ is a locally trivial \mathbb{A}^s -bundle in Zariski topology. For the tangent bundle T_X choose an isomorphism $\rho^* T_X \oplus E \cong \mathbf{1}_{\tilde{X}}^N$. Put

$$\mathrm{Th}(-T_X) = \Sigma_T^{-N} \Sigma_T^\infty \mathrm{Th}(E).$$

One can show that up to a canonical isomorphism $\mathrm{Th}(-T_X)$ does not depend on the choices made (see also [H05, Section 3]).

Theorem 5.3 ([H05, Theorem A.1]). *Let X be a smooth projective variety. Then $\Sigma_T^\infty X_+$ is strongly dualizable and there exists a canonical isomorphism*

$$(\Sigma_T^\infty X_+)^\vee \cong \mathrm{Th}(-T_X)$$

that gives rise to the evaluation isomorphism

$$[\mathbb{S}, \Sigma_T^\infty X_+(l)] \cong [\mathrm{Th}(-T_X), \mathbb{S}(l)].$$

Lemma 5.4. *In the notation of Definition 5.2 and Theorem 5.3 let $i_x: \operatorname{Spec} k \rightarrow X$ be a rational point. Then under the isomorphism*

$$[\mathbb{S}, \Sigma_T^\infty X_+] \cong [\operatorname{Th}(-T_X), \mathbb{S}]$$

morphism

$$\Sigma_T^\infty(i_x)_+: \mathbb{S} \rightarrow \Sigma_T^\infty X_+$$

corresponds to the composition

$$\begin{aligned} \operatorname{Th}(-T_X) &= \Sigma_T^{-N} \Sigma_T^\infty \operatorname{Th}(E) \rightarrow \Sigma_T^{-N} \Sigma_T^\infty E / (E - z(\tilde{x})) \xrightarrow{\sim} \\ &\xrightarrow{\sim} \Sigma_T^{-N} \Sigma_T^\infty \operatorname{Th}(E|_{z(\tilde{x})} \oplus N_{\tilde{x}/\tilde{X}}) \xrightarrow{\sim} \Sigma_T^{-N} \Sigma_T^\infty \operatorname{Th}(\mathbf{1}_{z(\tilde{x})}^N) \xrightarrow{\sim} \mathbb{S}. \end{aligned}$$

Here

- $z: \tilde{X} \rightarrow E$ is the zero section;
- $\tilde{x} = \rho^{-1}(x)$ for the rational point $x = i_x(\operatorname{Spec} k)$ and projection $\rho: \tilde{X} \rightarrow X$;
- the first morphism is given by the quotient map;
- the second morphism is given by the homotopy purity theorem;
- the third morphism is given by isomorphisms $N_{\tilde{x}/\tilde{X}} \cong (\rho^* T_X)|_{\tilde{x}}$ and $E \oplus \rho^* T_X \cong \mathbf{1}_{\tilde{X}}^N$,
- the last morphisms is given by projection $\mathbb{A}^s \cong \tilde{x} \rightarrow \operatorname{Spec} k$.

Proof. Follows from the proof of [H05, Theorem A.1]; see also (3.23) and (3.24) of loc. cit. \square

Lemma 5.5. *Let $\mathcal{X} \in \mathcal{SH}(k)_{t \geq 0}$ be a strongly dualizable object. Then $\mathcal{X}^\vee \in {}^\perp \mathcal{SH}(k)_{t \geq 1}$.*

Proof. For $A \in \mathcal{SH}(k)_{t \geq 1}$ we have

$$[\mathcal{X}^\vee, A] \cong [\mathbb{S}, \mathcal{X} \wedge A].$$

The last group vanishes since $\mathcal{X} \wedge A \in \mathcal{SH}(k)_{t \geq 1}$ by [ALP15, Lemma 4]. \square

Lemma 5.6. *For $\mathcal{Y} \in {}^\perp \mathcal{SH}(k)_{t \geq 1}$ and $A \in \mathcal{SH}(k)_{t \geq 0}$ the canonical morphism $A \rightarrow H_0^t(A)$ induces an isomorphism*

$$[\mathcal{Y}, A] \xrightarrow{\sim} [\mathcal{Y}, H_0^t(A)].$$

Proof. Consider the exact sequence

$$[\mathcal{Y}, A_{t \geq 1}] \rightarrow [\mathcal{Y}, A] \rightarrow [\mathcal{Y}, H_0^t(A)] \rightarrow [\mathcal{Y}, A_{t \geq 1}[1]]$$

associated to the triangle $A_{t \geq 1} \rightarrow A \rightarrow H_0^t(A) \rightarrow A_{t \geq 1}[1]$. Both the side terms are zero, whence the claim. \square

Corollary 5.7. *Let X be a smooth projective variety and $U \subset X$ be an open subvariety with $Z = X - U$ being smooth. Then for $A \in \mathcal{SH}(k)_{t \geq 0}$ the canonical morphism $A \rightarrow H_0^t(A)$ induces isomorphisms*

$$[(\Sigma_T^\infty X_+)^\vee, A] \xrightarrow{\sim} [(\Sigma_T^\infty X_+)^\vee, H_0^t(A)], \quad [(\Sigma_T^\infty U_+)^\vee, A] \xrightarrow{\sim} [(\Sigma_T^\infty U_+)^\vee, H_0^t(A)].$$

Proof. $\Sigma_T^\infty X_+$ and $\Sigma_T^\infty U_+$ are strongly dualizable by Theorem 5.3. Moreover, $\Sigma_T^\infty X_+, \Sigma_T^\infty U_+ \in \mathcal{SH}(k)_{t \geq 0}$ by [Mor04, Theorem 4.2.10]. The claim follows from Lemmas 5.5 and 5.6. \square

Recall the following cornerstone computation due to F. Morel.

Theorem 5.8 ([Mor04, Theorems 4.2.10 and 6.4.1] and [Mor12, Corollary 6.43]). $\mathbb{S} \in \mathcal{SH}_{t \geq 0}$ and $\pi_0^{\mathbb{A}^1}(\mathbb{S})_n \cong K_n^{\text{MW}}$.

Remark 5.9. There is another approach to the above computation given in [N14, Theorem 9.6] that is based on the theory of presheaves with framed transfers [GP14, GP15].

Theorem 5.10 (cf. [AH11, Theorem 4.3.1]). *Let X be a smooth projective variety of dimension d . Then there exist canonical isomorphisms*

$$[\mathbb{S}, \Sigma_T^\infty X_+(l)] \cong H^d(\mathcal{RS}_\bullet^X(X; \omega_X; H_0^t(\mathbb{S})(d+l))) \cong H_{\text{Nis}}^d(X; K_{d+l}^{\text{MW}} \otimes \omega_X) \cong H_{\text{Zar}}^d(X; K_{d+l}^{\text{MW}} \otimes \omega_X).$$

Here complex $\mathcal{RS}_\bullet^X(X; \omega_X; H_0^t(\mathbb{S})(d+l))$ looks as follows:

$$\begin{aligned} \bigoplus_{x \in X^{(0)}} K_{d+l}^{\text{MW}}(x; (\omega_X)|_x) &\rightarrow \bigoplus_{x \in X^{(1)}} K_{d+l-1}^{\text{MW}}(x; \Lambda_x^X \otimes (\omega_X)|_x) \rightarrow \dots \\ &\dots \rightarrow \bigoplus_{x \in X^{(d-1)}} K_{l+1}^{\text{MW}}(x; \Lambda_x^X \otimes (\omega_X)|_x) \rightarrow \bigoplus_{x \in X^{(d)}} K_l^{\text{MW}}(x). \end{aligned}$$

Let $i: \text{Spec } k \rightarrow X$ be a rational point and $l = 0$. Then under the above isomorphism morphism $\Sigma_T^\infty i_+ \in [\mathbb{S}, \Sigma_T^\infty X_+]$ corresponds to the cohomology class of $\langle 1 \rangle_x \in \mathcal{RS}_d^X(X; \omega_X; H_0^t(\mathbb{S})(d))$, where $\langle 1 \rangle_x \in \text{GW}(k) = K_0^{\text{MW}}(\text{Spec } k)$ belongs to the summand corresponding to $x = i(\text{Spec } k) \in X^{(d)}$.

Proof. Choose a Zariski locally trivial \mathbb{A}^s -bundle $\rho: \tilde{X} \rightarrow X$ and an isomorphism $\rho^*T_X \oplus E \cong \mathbf{1}_{\tilde{X}}^N$ as in Definition 5.2. We have the isomorphisms

$$[\mathbb{S}, \Sigma_T^\infty X_+(l)] \cong [\text{Th}(-T_X), \mathbb{S}(l)] \cong [\text{Th}(-T_X), H_0^t(\mathbb{S})(l)] \cong [\text{Th}(E), H_0^t(\mathbb{S})(N+l)[N]].$$

The first one is given by Theorem 5.3, the second one follows from Corollary 5.7 and the last one is given by suspension.

The first isomorphism of the theorem follows from Lemma 3.9 and Theorem 5.8. The identification of the cohomology of Rost-Schmid complex and the sheaf cohomology follows from Theorem 3.7. The identification of $\Sigma_T^\infty i_+$ follows from Lemma 5.4. \square

Theorem 5.11. *Let C_o be a smooth curve with a smooth compactification C and $D = C - C_o$. Then there are canonical isomorphisms*

$$\begin{aligned} [\mathbb{S}, \Sigma_T^\infty (C_o)_+(l)] &\cong H^1(\mathcal{RS}_\bullet^C(C, D; \omega_C; H_0^t(\mathbb{S})(l+1))) \cong \\ &\cong H_{\text{Nis}}^1(C; (K_{l+1}^{\text{MW}} \otimes \omega_C)_{C,D}) \cong H_{\text{Zar}}^1(C; (K_{l+1}^{\text{MW}} \otimes \omega_C)_{C,D}). \end{aligned}$$

Here

- $(K_{l+1}^{\text{MW}} \otimes \omega_C)_{C,D} = \ker \left[K_{l+1}^{\text{MW}} \otimes \omega_C \xrightarrow{i_D^*} (i_D)_*(K_{l+1}^{\text{MW}} \otimes i_D^* \omega_C) \right]$ with i_D being the closed embedding $i_D: D \rightarrow C$,
- complex $\mathcal{RS}_\bullet^C(C, D; \omega_C; H_0^t(\mathbb{S})(l+1))$ is consist of two terms

$$K_{l+1}^{\text{MW}}(\text{Spec } \mathcal{O}_{C,D}; (\omega_C)|_{\text{Spec } \mathcal{O}_{C,D}}) \rightarrow \bigoplus_{x \in C_o^{(1)}} K_l^{\text{MW}}(x) \oplus \bigoplus_{x \in D} K_{l+1}^{\text{MW}}(x; (\omega_C)|_x).$$

Let $i: \text{Spec } k \rightarrow C_o$ be a rational point and $l = 0$. Then under the above isomorphism morphism $\Sigma_T^\infty i_+ \in [\mathbb{S}, \Sigma_T^\infty (C_o)_+]$ corresponds to the cohomology class of $\langle 1 \rangle_x \in \mathcal{RS}_1^C(C, D; \omega_C; H_0^t(\mathbb{S})(1))$, where $\langle 1 \rangle_x \in \text{GW}(k) = K_0^{\text{MW}}(\text{Spec } k)$ belongs to the summand corresponding to $x = i(\text{Spec } k)$.

Proof. Choose a Zariski locally trivial \mathbb{A}^s -bundle $\rho: \tilde{C} \rightarrow C$ and an isomorphism $\rho^*T_C \oplus E \cong \mathbf{1}_{\tilde{C}}^N$ as in Definition 5.2.

Consider the triangle

$$\Sigma_T^\infty (C_o)_+ \rightarrow \Sigma_T^\infty C_+ \rightarrow \Sigma_T^\infty (C/C_o) \rightarrow \Sigma_T^\infty (C_o)_+[1].$$

Identifying $C/C_o \cong \text{Th}(N_{D/C}) = ((T_C)|_D)$ by the homotopy purity theorem [MV99, Theorem 2.23], applying $\underline{\text{Hom}}_{\mathcal{SH}(k)}(-, \mathbb{S})$ and rotating we obtain a triangle

$$(\Sigma_T^\infty \text{Th}((T_C)|_D))^\vee \rightarrow (\Sigma_T^\infty C_+)^\vee \rightarrow (\Sigma_T^\infty (C_o)_+)^\vee \rightarrow (\Sigma_T^\infty \text{Th}((T_C)|_D))^\vee[1].$$

Suspending with Σ_T^N and applying the isomorphism $(\Sigma_T^\infty C_+)^V \cong \mathrm{Th}(-T_C)$ from Theorem 5.3 we obtain a triangle

$$\Sigma_T^\infty \mathrm{Th}(E|_{\tilde{D}}) \rightarrow \Sigma_T^\infty \mathrm{Th}(E) \rightarrow \Sigma_T^N (\Sigma_T^\infty (C_o)_+)^V \rightarrow \Sigma_T^\infty \mathrm{Th}(E|_{\tilde{D}})[1].$$

Here $\tilde{D} = \rho^{-1}(D)$ and the identification $(\Sigma_T^\infty \mathrm{Th}((T_C)|_D))^V \cong \Sigma_T^\infty \mathrm{Th}(E|_{\tilde{D}})$ is given by [H05, Appendix A, Remark 1]. Moreover, it follows from loc. cit. that the first map in the triangle is induced by the embedding $D \rightarrow C$. Thus

$$C_o^V = (\Sigma_T^\infty (C_o)_+)^V \cong \Sigma_T^{-N} \Sigma_T^\infty (\mathrm{Th}(E)/\mathrm{Th}(E|_{\tilde{D}})).$$

We have the isomorphisms

$$[\mathbb{S}, \Sigma_T^\infty (C_o)_+(l)] \cong [C_o^V, \mathbb{S}(l)] \cong [C_o^V, H_0^t(\mathbb{S})(l)] \cong [\Sigma_T^\infty \mathrm{Th}(E)/\mathrm{Th}(E|_{\tilde{D}}), H_0^t(\mathbb{S})(N+l)[N]].$$

The first one is given by Theorem 5.3, the second one follows from Corollary 5.7 and the last one is given by the above considerations.

The first isomorphism of the theorem follows from Lemma 4.7 and Theorem 5.8. The identification of the cohomology of Rost-Schmid complex and the sheaf cohomology follows from Theorem 4.6 and an observation that for a closed point $x \in C$ the restriction homomorphism

$$K_{N+l}^{\mathrm{MW}}(\mathrm{Spec} \mathcal{O}_{C,x}) \rightarrow K_{N+l}^{\mathrm{MW}}(x)$$

is clearly surjective. The identification of $\Sigma_T^\infty i_+$ follows from Lemma 5.4. \square

Remark 5.12. One can show that

$$[\mathbb{S}, \Sigma_T^\infty (C_o)_+] \cong H^1(\mathcal{RS}_\bullet^C(C, D; \omega_C; H_0^t(\mathbb{S})(1)))$$

is canonically isomorphic to the cokernel

$$(K_1^{\mathrm{MW}})_{C,D}(\mathrm{Spec} \mathcal{O}_{C,D}; (\omega_C)|_{\mathrm{Spec} \mathcal{O}_{C,D}}) \xrightarrow{\partial} \bigoplus_{x \in C_o^{(1)}} \mathrm{GW}(k(x)).$$

This description is similar to the one given in [SV96, Lemma 2.3] for the relative Picard group.

APPENDIX A. SOME LEMMAS ABOUT PRESHEAVES WITH FRAMED TRANSFERS

In this section we follow the exposition of [GP14] and [GP15].

Definition A.1. Let X and Y be smooth varieties. An *explicit framed correspondence* Φ of level m consists of the following data:

- (1) a closed subset S in \mathbb{A}_X^m which is finite over X ;
- (2) an etale neighborhood $p: U \rightarrow \mathbb{A}_X^m$ of S ;
- (3) a collection of regular functions $\phi = (\phi_1, \phi_2, \dots, \phi_m)$ on U such that $S = \{\phi = 0\}$;
- (4) a regular morphism $U \rightarrow Y$.

Two explicit framed correspondences (S, U, ϕ, g) and (S', U', ϕ', g') of level m are said to be equivalent if $S = S'$ and there exists an etale neighborhood V of S in $U \times_{\mathbb{A}_X^m} U'$ such that $g \circ \pi_U = g' \circ \pi_{U'}$ and $\phi \circ \pi_U = \phi' \circ \pi_{U'}$ for the respective projections $\pi_U: V \rightarrow U$ and $\pi_{U'}: V \rightarrow U'$. The set of level m framed correspondences (i.e. explicit framed correspondences up to the above equivalence) is denoted $\mathrm{Fr}_m(X, Y)$. Note that $\mathrm{Fr}_0(X, Y) = \mathrm{Mor}_{\mathrm{Sm}_k}(X, Y)$ is the set of regular morphisms. Put

$$\mathbb{Z}\mathrm{F}_m(X, Y) = \mathbb{Z}[\mathrm{Fr}_m(X, Y)]/A,$$

where $\mathbb{Z}[\mathrm{Fr}_m(X, Y)]$ is the free abelian group on the set of level m framed correspondences and A is the subgroup generated by the elements

$$(S \sqcup S', U, \phi, g) - (S, U - S', \phi|_{U-S'}, g|_{U-S'}) - (S', U - S, \phi|_{U-S}, g|_{U-S}),$$

and denote $\mathbb{Z}\mathrm{F}_*(X, Y) = \bigoplus_{m \geq 0} \mathbb{Z}\mathrm{F}_m(X, Y)$.

Let X, Y and Z be smooth varieties and let $\Phi = (S, U, \phi, g) \in \text{Fr}_m(X, Y)$ and $\Psi = (T, W, \psi, h) \in \text{Fr}_l(Y, Z)$ be explicit correspondences. Then we compose them in the following way (see the details in [GP14]):

$$\Psi \circ \Phi = (S \times_Y T, U \times_Y W, (\phi \circ \pi_U, \psi \circ \pi_W), h \circ \pi_W).$$

One can show that this rule induces a composition $\mathbb{Z}\text{F}_m(X, Y) \times \mathbb{Z}\text{F}_l(Y, Z) \rightarrow \mathbb{Z}\text{F}_{m+l}(X, Z)$ that is associative. The *category of linear framed correspondences* $\mathbb{Z}\text{F}_*(k)$ has objects those of Sm_k and morphisms are the abelian groups $\mathbb{Z}\text{F}_*(X, Y)$. A *linear presheaf with framed transfers* is a contravariant functor from the category $\mathbb{Z}\text{F}_*(k)$ to the category of abelian groups.

We say that a linear presheaf with framed transfers \mathcal{F} is *homotopy invariant* if the canonical projection induces an isomorphism $\mathcal{F}(X) \cong \mathcal{F}(X \times \mathbb{A}^1)$ for every $X \in \text{Sm}_k$.

For every $X \in \text{Sm}_k$ denote

$$\sigma_X = (\{0\} \times X, \mathbb{A}^1 \times X, \pi_{\mathbb{A}^1}, \pi_X) \in \text{Fr}_1(X, X)$$

the suspension morphism. Here $\pi_{\mathbb{A}^1}$ and π_X are the respective projections. We say that a linear presheaf with framed transfers \mathcal{F} is *quasi-stable* if $\mathcal{F}(\sigma_X)$ is an isomorphism for every $X \in \text{Sm}_k$.

Definition A.2. Let X and Y be smooth varieties. An explicit framed correspondence $\Phi = (S, U, \phi, g)$ of level m from X to Y gives rise to a morphism of Nisnevich sheaves

$$\theta(\Phi): (\mathbb{P}^1, \infty)^{\wedge m} \wedge X_+ \rightarrow T^{\wedge m} \wedge Y_+$$

in the following way. Consider commutative diagram

$$\begin{array}{ccc} U - S & \xrightarrow{\quad} & U \\ \downarrow & & \downarrow p \\ (\mathbb{P}^1)^{\times m} \times X - S & \xrightarrow{i} & (\mathbb{P}^1)^{\times m} \times X \end{array} \quad \begin{array}{c} \searrow (\phi, g) \\ \searrow q \\ \rightarrow T^{\wedge m} \wedge Y_+ \cong \mathbb{A}_Y^n / ((\mathbb{A}^m - 0) \times Y) \end{array}$$

Here the square is cartesian, p is given by the composition $U \rightarrow \mathbb{A}_X^m \rightarrow (\mathbb{P}^1)^{\times m} \times X$ for the standard embedding $\mathbb{A}^1 = \mathbb{P}^1 - \infty \subset \mathbb{P}^1$, i is the open embedding and q is the constant morphism that maps $(\mathbb{P}^1)^{\times m}_X - S$ to the distinguished point. The square is a Nisnevich cover, thus we have a morphism of Nisnevich sheaves

$$((\mathbb{P}^1)^{\times m} \times X)_+ \rightarrow T^{\wedge m} \wedge Y_+$$

that induces a morphism

$$\theta(\Phi): (\mathbb{P}^1, \infty)^{\wedge m} \wedge X_+ \rightarrow T^{\wedge m} \wedge Y_+.$$

One can show [GP14, Lemma 5.2] that this rule gives a natural bijection

$$\theta: \text{Fr}_m(X, Y) \xrightarrow{\cong} \text{Map}_{\text{ShvNis}_\bullet}((\mathbb{P}^1, \infty)^{\wedge m} \wedge X_+, T^{\wedge m} \wedge Y_+).$$

For $A \in \mathcal{SH}(k)$ and $\Phi \in \text{Fr}_m(X, Y)$ let

$$\Phi^*: \pi_i^{\mathbb{A}^1}(A)_n(Y) \rightarrow \pi_i^{\mathbb{A}^1}(A)_n(X)$$

be given by the composition

$$\begin{aligned} \pi_i^{\mathbb{A}^1}(A)_n(Y) &= [Y[i], A(n)] \cong [T^{\wedge m} \wedge Y_+[i], T^{\wedge m} \wedge A(n)] \xrightarrow{\theta(\Phi)^*} \\ &\xrightarrow{\theta(\Phi)^*} [(\mathbb{P}^1, \infty)^{\wedge m} \wedge X_+[i], T^{\wedge m} \wedge A(n)] \cong [T^{\wedge m} \wedge X_+[i], T^{\wedge m} \wedge A(n)] \cong \\ &\cong [X[i], A(n)] = \pi_i^{\mathbb{A}^1}(A)_n(X). \end{aligned}$$

Here we used the canonical isomorphism $T \cong (\mathbb{P}^1, \infty)$ and suspension isomorphism Σ_T^m . One can check that this rule endows the presheaf of homotopy groups $\pi_i^{\mathbb{A}^1}(A)_n(-)$ with the structure of a *homotopy invariant quasi-stable linear presheaf with framed transfers* (see the discussion of radditive presheaves in [GP15, page 2]).

Lemma A.3 (cf. [MVW06, Theorem 22.2]). *Let \mathcal{F} be a homotopy invariant quasi-stable linear presheaf with framed transfers, e.g. $\mathcal{F} = \pi_i^{\mathbb{A}^1}(A)_n$ for $A \in \mathcal{SH}(k)$. Then the associated Zariski sheaf $\underline{\mathcal{F}}$ is a Nisnevich sheaf.*

Proof. By [GP15, Theorem 2.1] the associated Nisnevich sheaf $\underline{\mathcal{F}}^{\text{Nis}}$ has a canonical structure of a linear presheaf with framed transfers that is homotopy invariant and quasi-stable. Moreover, the associated morphism

$$\mathcal{F} \rightarrow \underline{\mathcal{F}}^{\text{Nis}}$$

is a morphism of linear presheaves with framed transfers. It is sufficient to check that for a smooth variety X and a point $x \in X$ the corresponding morphism $\mathcal{F}(\text{Spec } \mathcal{O}_{X,x}) \rightarrow \underline{\mathcal{F}}^{\text{Nis}}(\text{Spec } \mathcal{O}_{X,x})$ has trivial kernel and cokernel. Put $F = k(X)$ for the fraction field of $\mathcal{O}_{X,x}$ and consider the following diagram.

$$\begin{array}{ccccc} \mathcal{F}(\text{Spec } \mathcal{O}_{X,x}) & \longrightarrow & \underline{\mathcal{F}}^{\text{Nis}}(\text{Spec } \mathcal{O}_{X,x}) & \longrightarrow & (\underline{\mathcal{F}}^{\text{Nis}}/\mathcal{F})(\text{Spec } \mathcal{O}_{X,x}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(\text{Spec } F) & \xrightarrow{\cong} & \underline{\mathcal{F}}^{\text{Nis}}(\text{Spec } F) & \longrightarrow & (\underline{\mathcal{F}}^{\text{Nis}}/\mathcal{F})(\text{Spec } F) \end{array}$$

Here all the vertical morphisms are injective by [GP15, Theorem 2.15(3)]. Moreover, F is a field thus $\mathcal{F}(\text{Spec } F) \cong \underline{\mathcal{F}}^{\text{Nis}}(\text{Spec } F)$ and $(\underline{\mathcal{F}}^{\text{Nis}}/\mathcal{F})(\text{Spec } F) = 0$. The claim follows. \square

Lemma A.4. *Let \mathcal{F} be a homotopy invariant quasi-stable Nisnevich sheaf with framed transfers, e.g. $\mathcal{F} = \pi_i^{\mathbb{A}^1}(A)_n$ for $A \in \mathcal{SH}(k)$. Then for a smooth variety X and a finite collection of closed points $D \subset X^{(d)}$ we have*

$$H_{\text{Nis}}^1(\text{Spec } \mathcal{O}_{X,D}, \mathcal{F}) = 0.$$

Proof. By [GP15, Corollary 14.4 and Theorem 14.14] presheaf $H_{\text{Nis}}^1(-, \mathcal{F})$ has a canonical structure of a linear presheaf with framed transfers that is homotopy invariant and quasi-stable. Then

$$H_{\text{Nis}}^1(\text{Spec } \mathcal{O}_{X,D}, \mathcal{F}) \rightarrow H_{\text{Nis}}^1(\text{Spec } k(X), \mathcal{F}) = 0.$$

is injective by [GP15, Theorem 2.15(3)] (literally the same proof as in loc.cit. works for a semilocal scheme). \square

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